

Q -Meromorphic Functions, Quanta sets and Homomorphisms of the Quantum Plane

Ahmad Shafei Deh Abad

and

Vida Milani

*Institute for Studies in Theoretical Physics and Mathematics(IPM), Tehran,Iran.*¹

*Department of Mathematics, Faculty of Science, University of Tehran, Tehran,Iran.*²

Abstract

In this paper which is the completion of [1], we construct the $A_0(q)$ -algebra of Q -meromorphic functions on the quantum plane. This is the largest non-commutative, associative, $A_0(q)$ -algebra of functions constructed on the quantum plane. We also define the notion of quantum subsets of \mathbb{R}^2 which is a generalization of the notion of quantum disc and characterize some of their properties. In the end we study the Q -homomorphisms of the quantum plane.

¹Mailing address:Institute for Studies in Theoretical Physics and Mathematics(IPM),P.O.Box 19395-5746,Tehran,Iran.

E-mail:Shafei@netware2.ipm.ac.ir Milani@netware2.ipm.ac.ir

²Permanent address

1 Introduction

Non-commutative geometry and quantum groups are applied to problems of physics in different ways. In particular classical and quantum mechanics on the quantum plane have been studied in [2],[3] and [4]. As we have said in [1] our main objection is to transfer classical mechanics on a Poisson algebra \mathcal{B} to its functional quantization \mathcal{A} in the sense of [1]. More precisely we want to define an analogue of the Poisson bracket on \mathcal{A} and to develope an appropriate classical mechanics on \mathcal{A} parallel to that on \mathcal{B} . As a first step in this direction we gave a new interpretation of the Manin quantum plane in [1]. As a result the class of functions on the quantum plane was enlarged and the concept of Q -analytic functions was introduced. In this paper which is a complement of [1] we try to complete our necessary mathematical tools. It is worthwhile to note that our approach results in some new mathematical constructions interesting in themselves and which will be used in the formulation of classical mechanics and studying the integrability and non-integrability of Hamiltonian systems on quantum spaces.

In section 2 we enlarge the class of Q -analytic functions on the quantum plane and construct the $A_0(q)$ -algebra \mathcal{M}_Q of Q -meromorphic functions on the quantum plane. This algebra in one hand contains the $A_1(q)$ -algebra of Q -analytic functions on the quantum plane as its subalgebra, and on the other hand it can be considered as a quantization of the C-algebra of the absolutely convergent power series $\sum_{i,j>-1} a_{ij} t_1^i t_2^j$ on $\mathbb{R} \setminus \{0\} \times \mathbb{R} \setminus \{0\}$. The algebra \mathcal{M}_Q appears to be non-commutative, associative and having the unit element. Let $\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the permutation map. It will be clear that if $f \in \mathcal{M}_Q$ then $f \circ \sigma \in \mathcal{M}_Q$, and \mathcal{M}_Q with the above properties is maximal.

Section 3 is devoted to the study of the subsets of the quantum plane. In [1] we introduced some very special subsets of the quantum plane, i.e, quantum discs. Here we generalize this notion and define the quantum subsets of the quantum plane. A quantum subset Ω is identified by the algebra of Q -analytic functions on it. This algebra which is a non-commutative, associative $A_1(q)$ -algebra is in fact a functional quantization of a certain subalgebra of the commutative algebra of functions on Ω . If for a subset of \mathbb{R}^2 this subalgebra exists, then its functional quantization exists and is unique. The properties of the Q -analytic functions helps us to find some conditions for a subset of \mathbb{R}^2 to be a quantum set. Some of the properties of the quantum sets are also studied in this section. The algebra of functions on these quantum subsets is larger than that of the quantum plane and so the Hamiltonian systems on the quantum subsets has more chance to be integrable. This problem will also be true for the case of $\mathbb{R}-\{0\} \times \mathbb{R}-\{0\}$.

In section 4 the morphisms of the quantum plane are investigated. In fact these morphisms come from the homomorphisms of the $A_1(q)$ -algebra of Q -analytic functions on the quantum plane, exactly in the same way that smooth mappings of a manifold M into itself come from endomorphisms of $C^\infty(M)$ and vice versa. So it is natural to consider the homomorphisms of the $A_1(q)$ -algebra of Q -analytic functions on the quantum plane instead of its morphisms and by abuse of the language the homomorphisms of \mathcal{A}_Q will be called the homomorphisms of the quantum plane. Manin in his definition of $GL_q(2)$ uses the 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. All one knows about these matrices is that their elements satisfy some certain commutation relations.

In his approach these elements commute with the coordinate functions of the quantum plane x and p . Our approach is different, i.e., each automorphism of the quantum plane is determined

by a 2×2 matrix with elements in the $A_1(q)$ -algebra of Q -analytic functions on the quantum plane and so in general they do not commute with x and p . The only Q -analytic functions which commute with x and p are the elements of $A_1(q)$ which commute with each other and so can not generate the elements of the $GL_q(2)$ matrices.

Throughout this paper t_1 and t_2 will be the coordinate functions on the ordinary plane \mathbb{R}^2 . The same functions considered as coordinate functions on the quantum plane will be denoted respectively by x and p with the well-known commutation relation $px = qxp$. Moreover by the functional quantization of a C-algebra we mean its maximal $(1, D, A_1(q))$ or $(1, D - \{0\}, A_0(q))$ functional quantization in the sense of [1]. In the end it is worthwhile to say that the content of this paper and [1] can be generalized to any quantum space without any difficulty.

2 Q -Meromorphic Functions

Let $D = \{q \in \mathbb{C} : |q| \leq 1\}$ be the unit unit disc in \mathbb{C} . As in [1], $A_1(q)$ will be the C-algebra of all absolutely convergent power series $\sum_{i=0}^{\infty} a_i q^i$ in D with values in \mathbb{C} . Also we denote by $A_0(q)$ the C-algebra of all absolutely convergent power series $\sum_{i>-\infty} c_i q^i$ on $D - \{0\}$ with values in \mathbb{C} . We can generalize the concept of Q -analytic functions on the 2-intervals of \mathbb{R}^2 with values in $A_1(q)$ to the algebra of Q -analytic functions on the 2-intervals of \mathbb{R}^2 with values in $A_0(q)$ without any difficulty. Assume now that $\Omega = \mathbb{R} - \{0\} \times \mathbb{R} - \{0\}$ and let

$$f = \sum_{i>-\infty} \sum_{j,k>>-\infty} a_{ijk} q^i t_1^j t_2^k$$

be an absolutely convergent power series on $D - \{0\} \times \Omega$ with values in \mathbb{C} . (The sign $>>$ under the

second Σ indicates that j, k are bounded below). Clearly we can consider f as a function from

Ω into $A_0(q)$ admitting the absolutely convergent Laurent expansion

$$f = \Sigma_{i,j>>-\infty} a_{ij}(q) t_1^i t_2^j$$

on Ω . Since the above series is absolutely convergent on Ω , we can also write it as

$$f = \bar{\Sigma}_{i,j=0}^{\infty} t_1^{-i} \alpha_{ij}(t_1, t_2) t_2^{-j}$$

where the α_{ij} s are absolutely convergent power series on \mathbb{R}^2 with values in $A_0(q)$ and the sign -

over the Σ means that the indices are bounded above.

Definition 2.1. With the above notations and conventions let

$$\hat{f} = \bar{\Sigma}_{i,j=0}^{\infty} x^{-i} \hat{\alpha}_{ij}(x, p) p^{-j}$$

be obtained from f by the correspondence

$$t_1^i t_2^j = t_2^j t_1^i \rightarrow x^i p^j. \quad (+)$$

We call \hat{f} a *Q-meromorphic function* on Ω with values in $A_0(q)$ or simply a *Q-meromorphic*

function on Ω .

The two functions $\frac{1}{x}$ and $\frac{1}{p}$ are *Q-meromorphic* functions on Ω satisfying the following commutation relations

$$x \frac{1}{x} = \frac{1}{x} x = 1, p \frac{1}{p} = \frac{1}{p} p = 1$$

$$p \frac{1}{x} = q^{-1} \frac{1}{x} p, x \frac{1}{p} = q \frac{1}{p} x$$

$$\frac{1}{p} \cdot \frac{1}{x} = q \frac{1}{x} \cdot \frac{1}{p}, \left(\frac{1}{x}\right)^i = \frac{1}{x^i}, \left(\frac{1}{p}\right)^j = \frac{1}{p^j}.$$

By using these commutation relations we always follow the order $(x^i p^j)_{i,j>>-\infty}$ in writing the Q -meromorphic functions as above.

Remark. If $\hat{f}(x, p)$ is a Q -analytic function on the quantum plane with values in $A_0(q)$, then for $k, l \in \mathbb{Z}$, $\hat{f}(q^k x, q^l p)$ is a Q -analytic function on the quantum plane with values in $A_0(q)$.

Definition 2.2. The product of two Q -meromorphic functions

$$\hat{f} = \bar{\Sigma}_{i_1, i_2=0}^{\infty} x^{-i_1} \hat{a}_{i_1 i_2}(x, p) p^{-i_2}$$

$$\hat{g} = \bar{\Sigma}_{j_1, j_2=0}^{\infty} x^{-j_1} \hat{b}_{j_1 j_2}(x, p) p^{-j_2}$$

on Ω will be defined by

$$\hat{f} \cdot \hat{g} = \bar{\Sigma}_{i_1, i_2=0}^{\infty} \bar{\Sigma}_{j_1, j_2=0}^{\infty} q^{i_2 j_1} x^{-i_1 - j_1} (\hat{a}_{i_1 i_2}(x, q^{-j_1} p) \cdot \hat{b}_{j_1 j_2}(q^{-i_2} x, p)) p^{-i_2 - j_2}$$

where the above product between $\hat{a}_{i_1 i_2}$ and $\hat{b}_{j_1 j_2}$ is the product of two Q -analytic functions on the quantum plane with values in $A_0(q)$ in the sense of [1].

Lemma 2.1. With the above notations the product of two Q -meromorphic functions \hat{f} and \hat{g} on Ω is a Q -meromorphic function on Ω .

Proof. The proof is easily seen from the fact that $\hat{a}_{i_1 i_2}(x, q^{-j_1} p) \cdot \hat{b}_{j_1 j_2}(q^{-i_2} x, p)$ is a Q -analytic function on the quantum plane with values in $A_0(q)$.

From the above lemma we can see that the set of all Q -meromorphic functions on Ω with values in $A_0(q)$ is a non-commutative, associative $A_0(q)$ -algebra \mathcal{M}_Q with unity. This algebra contains \mathcal{A}_Q , the $A_1(q)$ -algebra of Q -analytic functions on the quantum plane with values in $A_1(q)$, as its subalgebra. It is clear that \mathcal{M}_Q is the $(1, D - 0, A_0(q))$ functional quantization of \mathcal{M} : the C-algebra of all absolutely convergent power series $\Sigma_{i,j>>-\infty} a_{ij} t_1^i t_2^j$ on Ω with values in

C , and if we denote by \mathcal{A} the C -algebra of all entire functions of the form $\sum_{i,j=0}^{\infty} a_{ij} t_1^i t_2^j$ on \mathbb{R}^2

with values in C , then the following diagram commutes

$$\Phi_{\mathcal{A}} : \mathcal{A}_Q \rightarrow \mathcal{A}$$

$$\downarrow \qquad \downarrow$$

$$\Phi_{\mathcal{M}} : \mathcal{M}_Q \rightarrow \mathcal{M}$$

where $\Phi_{\mathcal{A}}$ and $\Phi_{\mathcal{M}}$ are the quantization maps defined in [1] and $\mathcal{A}_Q \rightarrow \mathcal{M}_Q$ and $\mathcal{A} \rightarrow \mathcal{M}$ are the canonical injections.

3 Quantum Sets

Let D be the unit disc in C introduced in section 2, and let Ω be a non-empty subset of \mathbb{R}^2 .

Assume that

$$f = \sum_{i,j,k=0}^{\infty} a_{ijk} q^i t_1^j t_2^k$$

is an absolutely convergent power series on $D \times \Omega$ with values in C . We can consider f as

a function on Ω with values in $A_1(q)$ admitting the absolutely convergent Taylor expansion

$f = \sum_{i,j=0}^{\infty} a_{ij}(q) t_1^i t_2^j$ around $(0,0)$ on Ω . Let $\hat{f} = \sum_{i,j=0}^{\infty} a_{ij} x^i p^j$ be obtained from f by the correspondence (+) in section 2.

Definition 3.1. With the above notations \hat{f} is called a *Q-analytic function on the set Ω* with values in $A_1(q)$, or simply a *Q-analytic function on Ω* .

Let $\Omega \subseteq \mathbb{R}^2$ and $\mathcal{A}(\Omega)$ be the maximal C -algebra consisting of a) the C -algebra of all absolutely

convergent power series $\sum_{i,j=0}^{\infty} a_{ij} t_1^i t_2^j$ on \mathbb{R}^2 and b) for each $(\alpha, \beta) \in \mathbb{R}^2 - \Omega$ at least one analytic function on Ω admitting the absolutely convergent Taylor expansion $\sum_{i,j=0}^{\infty} a_{ij} t_1^i t_2^j$ on Ω with values in C having (α, β) as a singular point. Clearly $\mathcal{A}(\Omega)$ is a commutative, associative, C -algebra with unity.

Let Ω be a subset of \mathbb{R}^2 with non-empty interior, admitting the maximal C -algebra $\mathcal{A}(\Omega)$ with the above properties. Then the $(1, D, A_1(q))$ functional quantization of this algebra is seen to be the maximal $A_1(q)$ -algebra consisting of a) Q -analytic functions on the quantum plane with values in $A_1(q)$ and b) for each $(\alpha, \beta) \in \mathbb{R}^2 - \Omega$ at least one Q -analytic function on Ω with values in $A_1(q)$ which is not defined at (α, β) . This functional quantization is denoted by $\mathcal{A}_Q(\Omega)$ and is clearly an associative, non-commutative, $A_1(q)$ -algebra with unity. It is also clear that if for $\Omega \subseteq \mathbb{R}^2$, $\mathcal{A}(\Omega)$ exists then $\mathcal{A}_Q(\Omega)$ exists and is unique.

Definition 3.2. With the above notations and conventions, $\mathcal{A}_Q(\Omega)$ is called *the algebra of Q -analytic functions on the set Ω* and the pair $(\Omega, \mathcal{A}_Q(\Omega))$ or simply Ω is called a *quantum subset* of \mathbb{R}^2 or simply a *quantum set*.

The following examples show that there exist some subsets of \mathbb{R}^2 which are quantum sets. The first example indicates that the definition of the quantum set is in fact a generalization of the definition of the quantum disc defined in [1].

Example 3.1. let $\Omega \subseteq \mathbb{R}^2$ be a disc with center $(0,0)$. We remind from [1] that for each $(\alpha, \beta) \in \mathbb{R}^2 - \Omega$, if we set

$$f_{\alpha, \beta} = \frac{1}{t_1^2 + t_2^2 - (\alpha^2 + \beta^2)} = \sum_{i,j=0}^{\infty} a_{ij} t_1^i t_2^j,$$

then $\hat{f}_{\alpha, \beta} = \sum_{i,j=0}^{\infty} a_{ij} x^i p^j$ is a Q -analytic function on Ω and can not be defined at (α, β) . So

there is a maximal $A_1(q)$ -algebra $\mathcal{A}_Q(\Omega)$ containing the $A_1(q)$ -algebra generated by \mathcal{A}_Q and $\hat{f}_{\alpha,\beta}$

for each $(\alpha, \beta) \in \mathbb{R}^2 - \Omega$. Therefore Ω is a quantum set.

Example 3.2. Let $\Omega \subseteq \mathbb{R}^2$ be the interior of a closed curve in \mathbb{R}^2 given by the polynomial equation $f(t_1, t_2) = 0$. Assume that $(0, 0) \in \Omega^o$ (the interior of Ω) and Ω is symmetric with respect to the t_1, t_2 axis in \mathbb{R}^2 . For each $(c, d) \in \mathbb{R}^2 - \Omega$ let (α, β) be the intersection of $\delta\Omega$ with the line passing through $(0,0)$ and (c, d) and let $r = \frac{\sqrt{\alpha^2 + \beta^2}}{\sqrt{c^2 + d^2}}$. Set

$$g(t_1, t_2) = \frac{1}{f(rt_1, rt_2)} = \sum_{i,j=0}^{\infty} a_{ij} t_1^i t_2^j$$

then $\hat{g}(x, p) = \sum_{i,j=0}^{\infty} a_{ij} x^i p^j$ is a Q -analytic function on Ω and is not defined at (c, d) . It follows that Ω is a quantum set.

The following examples will be used later in this section.

Example 3.3. Let $\Omega \subseteq \mathbb{R}^2$ be the polygone obtained from the intersection of the two bands

$$|t_1 + t_2| < 1, |t_1 - t_2| < 1.$$

For each $(\alpha, \beta) \in \mathbb{R}^2 - \Omega$, let (a, b) be the intersection of $\delta\Omega$ (the boundary of Ω) with the line passing through $(0,0)$ and (α, β) and let $r = \frac{\sqrt{\alpha^2 + \beta^2}}{\sqrt{a^2 + b^2}}$. Now if

$$f_{\alpha,\beta} = \frac{1}{(t_1 + t_2)^2 - r^2} \cdot \frac{1}{(t_1 - t_2)^2 - r^2} = \sum_{i,j=0}^{\infty} a_{ij} t_1^i t_2^j,$$

then $\hat{f}_{\alpha,\beta}(x, p) = \sum_{i,j=0}^{\infty} a_{ij} x^i p^j$ is a Q -analytic function on Ω and is not defined at (α, β) . So there is a maximal $A_1(q)$ -algebra $\mathcal{A}_Q(\Omega)$ generated by \mathcal{A}_Q and $\hat{f}_{\alpha,\beta}$ for $(\alpha, \beta) \in \mathbb{R}^2 - \Omega$. So Ω is a quantum set.

Example 3.4. Let $a \in \mathbb{R}$ and $\Omega = \mathbb{R} \times (-a, a)$. Then for $(\alpha, \beta) \in \mathbb{R}^2 - \Omega$ if

$$f_{\alpha,\beta} = \frac{1}{t_2^2 - \beta^2} = \sum_{i=0}^{\infty} a_i t_2^i$$

then $\hat{f}_{\alpha,\beta} = \sum_{i=0}^{\infty} a_i p^i$ is a Q -analytic function on Ω and is not defined at (α, β) . So Ω is a quantum set and we call it the *quantum horizontal band*. In the same way we see that $\Theta = (-a, a) \times \mathbb{R}$ is a quantum set which is called the *quantum vertical band*.

Proposition 3.1. If the intersection of a family of quantum sets has a non-empty interior, then it is a quantum set.

Proof. Let $(\Omega_i, \mathcal{A}_Q(\Omega_i))_{i \in I}$ be a family of quantum sets. Then every Q -analytic function on Ω_i can be considered as a Q -analytic function on the intersection $\bigcap_{i \in I} \Omega_i$. With this hypothesis let \mathcal{A} be the $A_1(q)$ -algebra generated by $\bigcup_{i \in I} \mathcal{A}_Q(\Omega_i)$. Let $(\alpha, \beta) \in \mathbb{R}^2 - \bigcap_{i \in I} \Omega_i$, then for at least one $i \in I$, $(\alpha, \beta) \in \mathbb{R}^2 - \Omega_i$, and so there is a maximal $A_1(q)$ -algebra containing \mathcal{A} and consisting of \mathcal{A}_Q and for each $(\alpha, \beta) \in \mathbb{R}^2 - \bigcap_{i \in I} \Omega_i$ at least one Q -analytic function on $\bigcap_{i \in I} \Omega_i$ with singularity at (α, β) . This maximal $A_1(q)$ -algebra is the algebra of Q -analytic functions on $\bigcap_{i \in I} \Omega_i$, i.e., $\mathcal{A}_Q(\bigcap_{i \in I} \Omega_i)$. So the proof is complete.

Corollary. The open or closed 2-intervals in \mathbb{R}^2 are quantum sets.

Proposition 3.2. The closure of a quantum set is a quantum set.

Proof. Let Ω be a quantum set with non-empty boundary. Let $(\alpha, \beta) \in \mathbb{R}^2 - \bar{\Omega}$ (where $\bar{\Omega}$ is the closure of Ω). Let (c, d) be the intersection of $\delta\Omega$ with the line passing through $(0,0)$ and (α, β) . There exists a point $(a, b) \in \mathbb{R}^2 - \bar{\Omega}$ on this line between (c, d) and (α, β) and a Q -analytic function \hat{f} on Ω which is not defined at (α, β) . Now let $r = \frac{\sqrt{a^2+b^2}}{\sqrt{\alpha^2+\beta^2}}$, then the function $\hat{g}_{\alpha,\beta}(x, p) = \hat{f}(rx, rp)$ is a Q -analytic function on $\bar{\Omega}$ and is not defined at (α, β) . So there exists a maximal $A_1(q)$ -algebra containing the $A_1(q)$ -algebra generated by \mathcal{A}_Q and all the $\hat{g}_{\alpha,\beta}$ s for each $(\alpha, \beta) \in \mathbb{R}^2 - \bar{\Omega}$. this maximal algebra is the algebra of Q -analytic functions on $\bar{\Omega}$. So $\bar{\Omega}$ is a

quantum set.

Lemma 3.1. Let $\Omega \subseteq \mathbb{R}^2$ and \hat{f} be a Q -analytic function on Ω . Then for each $(\alpha, \beta) \in \Omega$, \hat{f}

is a Q -analytic function on the open open 2-interval $(-\alpha, \alpha) \times (-\beta, \beta) \subseteq \mathbb{R}^2$.

Proof. Let $\hat{f} = \sum_{i,j=0}^{\infty} a_{ij} x^i p^j$. Then the series $\sum_{i,j=0}^{\infty} a_{ij} t_1^i t_2^j$ is absolutely convergent on the open 2-interval $(-\alpha, \alpha) \times (-\beta, \beta) \subseteq \mathbb{R}^2$. The proof is complete.

Corollary. If $\Omega \subseteq \mathbb{R}^2$ is a quantum set, then for $(\alpha, \beta) \in \Omega$, the open 2-interval $(-\alpha, \alpha) \times (-\beta, \beta)$ is contained in Ω . Consequently every quantum set contains the origin and is symmetric with respect to the coordinate axis.

Proposition 3.4. Every convex set $\Omega \subseteq \mathbb{R}^2$ containing $(0,0)$ and symmetric with respect to the coordinate axis is a quantum set if its interior Ω^o is non-empty.

Proof. If $\Omega = \mathbb{R}^2$, then clearly it is a quantum set. So let $\Omega \subset \mathbb{R}^2$. Since Ω is convex then for each $(\alpha, \beta) \in \mathbb{R}^2 - \Omega$ there is a line passing through (α, β) and having empty intersection with Ω .

If Ω is not bounded since it contains the origin and is symmetric with respect to the coordinate axis then for each $(\alpha, \beta) \in \mathbb{R}^2 - \Omega$ there is a band $(-\alpha, \alpha) \times \mathbb{R}$ (or $\mathbb{R} \times (-\beta, \beta)$) containing Ω . Let

$f = \frac{1}{t_1^2 - \alpha^2} = \sum_{i=0}^{\infty} a_i t_1^i$ (or $f = \frac{1}{t_2^2 - \beta^2} = \sum_{i=0}^{\infty} a_i t_2^i$), then $\hat{f}_{\alpha, \beta} = \sum_{i=0}^{\infty} a_i x^i$ (or $\hat{f}_{\alpha, \beta} = \sum_{i=0}^{\infty} a_i p^i$) is a Q -analytic function on Ω and is not defined at (α, β) . [See example 3.4]. If Ω is bounded then from the assumptions on Ω it follows that for $(\alpha, \beta) \in \mathbb{R}^2 - \Omega$ there is a subset of \mathbb{R}^2 with equation $|t_1 + t_2| < r, |t_1 - t_2| < r$ (or $(-\alpha, \alpha) \times (-\beta, \beta)$) containing Ω and having the point (α, β) on its boundary. Let

$$f = \frac{1}{(t_1 + t_2)^2 - r^2} \cdot \frac{1}{(t_1 - t_2)^2 - r^2} = \sum_{i,j=0}^{\infty} a_{ij} t_1^i t_2^j$$

(or $f = \frac{1}{t_1^2 - \alpha^2} \cdot \frac{1}{t_2^2 - \beta^2} = \sum_{i,j=0}^{\infty} a_{ij} t_1^i t_2^j$), then $\hat{f}_{\alpha, \beta} = \sum_{i,j=0}^{\infty} a_{ij} x^i p^j$ is a Q -analytic function on Ω and

is not defined at (α, β) [see examples 3.3,3.4]. In each case there is a maximal $A_1(q)$ -algebra containing the $A_1(q)$ -algebra generated by \mathcal{A}_Q and $\hat{f}_{\alpha, \beta}$ for each $(\alpha, \beta) \in \mathbb{R}^2 - \Omega$. This maximal $A_1(q)$ -algebra is the algebra of Q -analytic functions on Ω . The proof is complete.

The following example shows that not all the quantum sets are convex.

Example 3.5 Let $\Omega \subseteq \mathbb{R}^2$ be the intersection of two sets given by the equations

$$t_1 t_2 < 1, t_1 t_2 < -1,$$

then for each $(\alpha, \beta) \in \mathbb{R}^2 - \Omega$ let $(a, b) \in \delta\Omega$ be the intersection of $\delta\Omega$ with the line passing through $(0,0)$ and (α, β) . Let Ω' be obtained from Ω by the coorespondence

$$\Omega \rightarrow \Omega'$$

$$(t_1, t_2) \mapsto \frac{\sqrt{a^2 + b^2}}{\sqrt{\alpha^2 + \beta^2}}(t_1, t_2).$$

Let $\lambda = \frac{\sqrt{\alpha^2 + \beta^2}}{\sqrt{a^2 + b^2}}$ and

$$f = \frac{1}{t_1 t_2 - \lambda^2} \cdot \frac{1}{t_1 t_2 + \lambda^2} = \sum_{i,j=0}^{\infty} a_{ij} t_1^i t_2^j$$

then $\hat{f}_{\alpha, \beta} = \sum_{i,j=0}^{\infty} a_{ij} x^i p^j$ is a Q -analytic function on Ω and is not defined at (α, β) . So there is a maximal $A_1(q)$ -algebra containing the $A_1(q)$ -algebra generated by \mathcal{A}_Q and $\hat{f}_{\alpha, \beta}$ for each $(\alpha, \beta) \in \mathbb{R}^2 - \Omega$. This maximal algebra is the algebra of Q -analytic functions on Ω .

Proposition 3.5. Every quantum set Ω is contractible.

Proof. From lemma 3.1 it is clear that

$$H : [0, 1] \times \Omega \rightarrow \Omega$$

$$(t, \omega) \mapsto t\omega$$

for $t \in [0, 1]$ and $\omega \in \Omega$ is the contraction map.

4 Homomorphisms of the Quantum Plane

Definition 4.1. An $A_1(q)$ -homomorphism $\Theta : \mathcal{A}_Q \rightarrow \mathcal{A}_Q$ is called a *homomorphism* of the quantum plane.

Remark. In order to determine a homomorphism Θ of the quantum plane it is sufficient to define $\Theta(x)$ and $\Theta(p)$ and then extend it by

$$\Theta(\sum_{i,j=0}^{\infty} a_{ij} x^i p^j) = \sum_{i,j=0}^{\infty} a_{ij} \Theta(x)^i \Theta(p)^j$$

to the whole \mathcal{A}_Q . Now let $\Theta : \mathcal{A}_Q \rightarrow \mathcal{A}_Q$ be a homomorphism of the quantum plane such that $\Theta(x) \neq 0$ and $\Theta(p) \neq 0$. Since

$$\Theta(p)\Theta(x) = q\Theta(x)\Theta(p)$$

there are Q -analytic functions f, g, h, k on the quantum plane such that

$$\Theta(x) = xf + gp$$

$$\Theta(p) = xh + kp$$

and the following relation satisfies:

$$\begin{aligned} & x^2[h(x, qp)f(x, p) - qf(x, qp)h(x, p)] + \\ & xp[h(q^{-1}x, p)g(q^{-1}x, p) - q^2g(q^{-1}x, qp)h(x, p) + qk(q^{-1}x, qp)f(x, p) - qf(x, p)k(q^{-1}x, p)] + \\ & p^2[k(q^{-2}x, p)g(q^{-1}x, p) - qg(q^{-2}x, p)k(q^{-1}x, p)] = 0. \end{aligned} \tag{*}$$

The set of all homomorphisms of the quantum plane is denoted by $Hom_Q(\mathbb{R}^2)$.

Definition 4.2. The invertible elements of $Hom_Q(\mathbb{R}^2)$ are called the *automorphisms* of the quantum plane.

Remark. If $\Theta : \mathcal{A}_Q \rightarrow \mathcal{A}_Q$ is an automorphism of the quantum plane and

$$\Theta(x) = xf + gp, \Theta(p) = xh + kp$$

then there are Q -analytic functions f_1, h_1 on the quantum plane such that we can write

$$\Theta(x) = f_1x + gp, \Theta(p) = h_1x + kp.$$

So we can correspond a 2×2 matrix to Θ by

$$\begin{pmatrix} f_1 & g \\ h_1 & k \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} \Theta(x) \\ \Theta(p) \end{pmatrix}.$$

The set of automorphisms of the quantum plane with the composition rule is a group. But in the above matrix form if Θ_1, Θ_2 are two automorphisms of the quantum plane then the product of their corresponding matrices does not necessarily correspond to an automorphism of the quantum plane. To see this, let

$$\begin{pmatrix} \Theta_1(x) \\ \Theta_1(p) \end{pmatrix} = \begin{pmatrix} e^p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix},$$

$$\begin{pmatrix} \Theta_2(x) \\ \Theta_2(p) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^x \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}.$$

Then

$$\begin{pmatrix} e^p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^x \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} e^p & 0 \\ 0 & e^x \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}.$$

But the elements of the product matrix do not satisfy (*), since

$$e^{qp}.e^x \neq e^{q^{-1}x}.e^p.$$

But in the limit $q \rightarrow 1$, the product of matrices corresponding to the automorphisms will correspond to an automorphism. So clearly we can consider the set of matrices corresponding to the automorphisms of $\mathcal{A}_Q(\mathbb{R}^2)$ as a functional quantization of the group of automorphisms of $\mathcal{A}(\mathbb{R}^2)$ in the sense that the elements of the matrices are functional quantizations of the elements of the matrices corresponding to the automorphisms of $\mathcal{A}(\mathbb{R}^2)$.

References

- [1] A. Shafei Deh Abad and V. Milani, *J.M.P*, vol.35, no.9, 5074-5086.
- [2] S. V. Shabanov, *J.Phys.A:Math.Gen.* 26 (1993) 2583-2606.
- [3] I. Ya. Arefeva and I. V. Volovich, *Phys.letter B*, 268 (1991) 179-187.
- [4] J. Schwenk and J. Wess, *Phys.letter B*, 291 (1992) 273-277.

Acknowledgements

The authors would like to thank Professor Abdus Salam, The International Atomic Energy Agency and Prof. Narasimhan for hospitality at the International Centre for Theoretical Physics, Trieste, Italy where part of this work has been done.